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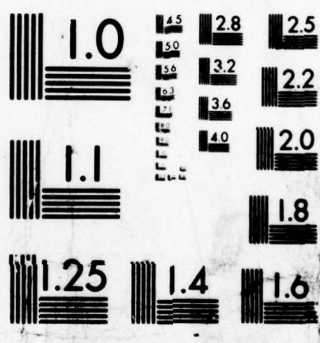

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CONSISTENT ESTIMATION OF CONTINUOUS-TIME SIGNALS FROM QUANTIZED NOISY SAMPLES

by

Elias Masry and Stamatis Cambanis

Abstract

It is well known that a continuous-time signal $f(t)$, $-\infty<t<\infty$, cannot be reconstructed from its 2-level quantized version $\mathrm{sgn}[f(t)]$. It is shown that by deliberately corrupting equally-spaced samples $\{f(k/W)\}$ of f by additive Gaussian noise $\{\xi_k\}$ before hardlimiting, the signal f can be estimated consistently from the binary sequence $\{\mathrm{sgn}[f(k/W) + \xi_k]\}$ as the sampling rate $W\to\infty$. A class of nonlinear estimates is introduced and bounds on the mean-square error are obtained. The signal f need not be bandlimited.

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I. INTRODUCTION

Let $f(t)$, $-\infty<t<\infty$, be a real continuous function. It is well known that, in general, f cannot be determined from $\mathrm{sgn}[f(t)]$. This situation remains true even when f is analytic, such as a bandlimited function [1]. We recall that for a bandlimited function

$$(1)\quad f(t) = \int_{-W}^{W} e^{it\lambda} F(\lambda)d\lambda, \quad F(\lambda)\in L_1[-W,W],$$

we have by Titchmarsh [1] that f can be represented by the conditionally convergent series

$$(2)\quad f(t) = f(0) \prod_{n=1}^{\infty} \left(1 - \frac{t}{z_n}\right)$$

where $f(0)\neq 0$ and $\{z_n\}$ is the set of all (real and complex) zeros of $f(z)$, $z = t+iu$, in the complex plane. Thus f is determined, up to a multiplicative constant, by its real <u>and</u> complex zeros; the complex zeros, however, are not observable. Duffin and Schaeffer [2] have shown in 1938 that by subtracting a cosine function $C \cos Wt$ from f, the resulting function has real zeros only. Unaware of Duffin and Schaeffer's result, Bar-David [3] reproduced a weaker version of it. Duffin and Schaeffer's result is given by

<u>Proposition 1 [2]</u>. Let $f(z)$, $z = t+iu$, be an entire function of exponential type with exponent W, $f(z) = O(e^{W|z|})$, such that $|f(t)| \leq 1$. Then the function $g(z) \triangleq C \cos Wz - f(z)$, $C > 1$, has real simple zeros only.

Substituting z for t in (1), we have an entire function of exponential type with exponent W and $|f(t)|\leq A$, $A = \int_{-W}^{W} |F(\lambda)|d\lambda$. It follows by Proposition 1 that for a bandlimited function f given by (1) we have, with $C>A$, that $g(t) = C \cos Wt-f(t)$ has real zeros $\{t_k\}$ only and

$$g(t) = g(0) \prod_{k=1}^{\infty} \left(1 - \frac{t}{t_k}\right) = [C - f(0)] \prod_{k=1}^{\infty} \left(1 - \frac{t}{t_k}\right)$$

so that

$$(3) \quad f(t) = C \cos Wt - [C-f(0)] \prod_{k=1}^{\infty} \left(1 - \frac{t}{t_k}\right) .$$

The practical significance of (3) for digital transmission of continuous-time signals f is doubtful, since (i) a synchronous tone C cos Wt is needed at the receiver and (ii) no <u>digital</u> reconstruction scheme of f based on sgn[g(t)] is available.

A new approach is presented in this paper. We do <u>not</u> assume that f is bandlimited, and we <u>do</u> reconstruct f from the sign of <u>deliberately</u> corrupted time-samples of f, as the sampling rate tends to infinity. The approach is motivated by the results of a recent paper [4] by the present authors.

II. NEW RECONSTRUCTION SCHEME

Consider the diagram depicted in Figure 1. A continuous-time signal f is sampled at equally-spaced points $\{\frac{k}{W_n}\}_{k=-\infty}^{\infty}$, where W_n is the $n^{\underline{th}}$ sampling rate, $W_n \uparrow \infty$ as $n \to \infty$. Each sample $f(\frac{k}{W_n})$ is deliberately corrupted by an additive Gaussian variate $\xi_{n,k}$, i.e.,

$$(4) \quad Y_{n,k} = f(\frac{k}{W_n}) + \xi_{n,k} \qquad k = \ldots,-1,0,1,\ldots$$

where for each fixed n, $\{\xi_{n,k}\}_{k=-\infty}^{\infty}$ is a sequence of independent identically distributed Gaussian random variables with means zero and variances σ^2. Only the sign of $\{Y_{n,k}\}_{k=-\infty}^{\infty}$ is transmitted, i.e.,

$$(5) \quad Z_{n,k} = \mathrm{sgn}[Y_{n,k}] = \mathrm{sgn}[f(\frac{k}{W_n}) + \xi_{n,k}], \quad k = \ldots,-1,0,1\ldots .$$

At the receiver, the binary sequence $\{Z_{n,k}\}_{k=-\infty}^{\infty}$ is used to obtain a consistent estimate $\hat{f}_n(t)$ of f(t), i.e., $\hat{f}_n(t)$ converges to f(t) in quadratic mean as the sampling rate $W_n \to \infty$. Note that without the additive noise $\{\xi_{n,k}\}_{k=-\infty}^{\infty}$, it is not possible to reconstruct f(t) from $\{\mathrm{sgn}[f(\frac{k}{W_n})]\}_{k=-\infty}^{\infty}$ as $n \to \infty$, i.e., from sgn[f(t)], $-\infty<t<\infty$.

We shall provide reconstruction procedures for the following class of signals f.

Definition . Let $U(c_0)$ be the class of real bounded uniformly continuous functions f on a finite or infinite interval $I = [a,b]$, $-\infty \le a<b \le \infty$, such that $|f(t)| \le c_0$ for all $t \in I$.

The structure of the receiver is as follows: Let

$$(6) \quad \mu(x) = \sqrt{\frac{2}{\pi}} \int_0^{x/\sigma} e^{-u^2/2} du, \qquad -\infty<x<\infty,$$

and note that μ is odd and strictly monotonic on $(-\infty,\infty)$. Let

$$(7) \quad \hat{m}_n(t) = \sum_{k=-\infty}^{\infty} Z_{n,k}\, h_k(n,t) \ , \quad t \in I$$

where $\{h_k(n,t)\}_{k=-\infty}^{\infty}$ is a sequence of positive kernels to be specified below. Let $c = c_0 + \eta$, $\eta > 0$, and set

$$(8) \qquad \hat{f}_n(t) = \begin{cases} \mu^{-1}[\hat{m}_n(t)] & , \ |\hat{m}_n(t)| \le \mu(c) \\ 0 & \ \ |\hat{m}_n(t)| > \mu(c) \end{cases} .$$

The proofs of the following results can be found in [5].

<u>Theorem 1</u>. Assume the kernels $\{h_k(n,t)\}_{k=-\infty}^{\infty}$ satisfy for each $t \in I$

i. $\sum_{k=-\infty}^{\infty} h_k(n,t) = 1 \quad , \quad n = 1,2,\ldots$

ii. $\sum_{k=-\infty}^{\infty} (t - \frac{k}{W_n})^2\, h_k(n,t) \to 0$ as $n \to \infty$

iii. $\sum_{k=-\infty}^{\infty} h_k^2(n,t) \to 0$ as $n \to \infty$.

Then for every $f \in U(c_0)$ we have $\hat{f}_n(t) \to f(t)$, in quadratic mean as $n \to \infty$, for each $t \in I$.

We now consider two specific sequences of kernels $\{h_k(n,t)\}_{k=-\infty}^{\infty}$ and obtain bounds on the mean square error. More general kernels are discussed in [5]. We have

$$(9a) \quad h_k(n,t) = \frac{(W_n t)^k e^{-W_n t}}{k!} \ , \quad k = 0,1,\ldots, \quad t \in I = [0,\infty)$$

for Szasz's interpolation kernel, and

$$(9b) \quad h_k(n,t) = \binom{W_n}{k} t^k (1-t)^{W_n - k}, \quad k = 0,1,\ldots,W_n \qquad t \in I = [0,1]$$

For Ber nstein's interpolation kernel. (Here W_n is an integer.)

__Theorem 2.__ Let $f \in \mathcal{U}(c_0)$ and $\hat{f}_n(t)$ be given by (8) with Szasz's interpolation kernel (9a). Then for every $t \in [0,\infty)$

$$E[\hat{f}_n(t) - f(t)]^2 \leq A_1(c)\, \omega^2(f, \sqrt{t/W_n}\,) + A_2(c)\, e^{-2W_n t}\, I_0(2W_n t) \quad .$$

where $\omega(f,\delta)$ is the modulus of continuity of f over $[0,\infty)$, $I_0(x)$ is the modified Bessel function of the first kind of order zero and $A_i(c)$, $i = 1,2$, are constants independent of n (depending on c and σ^2 only).

__Corollary.__ Under the assumptions of Theorem 2,

a. $\hat{f}_n(t) \to f(t)$ in quadratic mean as $n \to \infty$ uniformly on compact subsets of $(0,\infty)$.

b. If $f \in \mathrm{Lip}\ \alpha$, $0 < \alpha \leq 1$, then

$$E[\hat{f}_n(t) - f(t)]^2 = \mathcal{O}(W_n^{-\min(1/2,\alpha)})$$

uniformly on compact subsets of $(0,\infty)$.

c. If $f \in \mathrm{Lip}\ 1$ and $W_n = \mathcal{O}(n^a)$, $a > 2$, then $\hat{f}_n(t)$ converges to $f(t)$ with probability one as $n \to \infty$, uniformly on compact subsets of $(0,\infty)$.

We remark that the variance σ^2 of the Gaussian variates $\{\xi_{n,k}\}$ is completely arbitrary. It should be clear on intuitive grounds that as the bound c_0 of f increases, so must σ^2. It is seen from Theorem 2 that σ^2 only affects the values of the constants $A_i(c)$, $i = 1,2$. It turns out that a nearly optimal choice for σ is $\sigma = c = c_0 + \eta$ for which

$$A_1(c) = \frac{8}{\pi c^2} A_2(c)$$

$$A_2(c) = [\frac{\pi e c^2}{2} + \frac{2}{\pi e \eta^2}]$$

Similar results can be obtained for the estimate (8) with Ber nstein's kernel (9b). We have

<u>Theorem 3</u>. Let f be continuous on $[0,1]$ such that $|f(t)| \leq c_0$ for all $t \in [0,1]$. Then for every $t \in [0,1]$ the estimate (8) with Ber nstein's kernel (9b) satisfies

$$E[\hat{f}_n(t) - f(t)]^2 \leq A_1(c)\, \omega^2(f, \sqrt{t(1-t)/W_n}\,) + A_2(c)[W_n t(1-t)]^{-1/2}(1+o(1))$$

where $\omega(f,\delta)$ is the modulus of continuity of f over $[0,1]$ and $A_i(c)$, $i = 1,2$, are constants independent of n.

<u>Corollary</u>. Under the assumptions of Theorem 3,

a. $\hat{f}_n(t) \to f(t)$ in quadratic mean as $n \to \infty$ uniformly on $[a,b] \subset (0,1)$.

b. If $f \in \mathrm{Lip}\,\alpha$, $0 < \alpha \leq 1$, then

$$E[\hat{f}_n(t) - f(t)]^2 = O(W_n^{-\min(1/2,\alpha)}) \text{ uniformly on } [a,b] \subset (0,1).$$

c. If $f \in \mathrm{Lip}\,1$ and $W_n = O(n^a)$, $a>2$, then $\hat{f}_n(t) \to f(t)$ with probability one as $n \to \infty$, uniformly on $[a,b] \subset (0,1)$.

We remark that the interval $[0,1]$ in Theorem 3 can be replaced by any <u>finite</u> interval $[a,b]$ by proper scaling.

We also note that the Gaussian density of the variates $\{\xi_{n,k}\}$ in (4) can be replaced by other symmetric densities which are positive over $(-\infty,\infty)$. For example, one could use Laplacian density $\phi(x) = \frac{\alpha}{2} e^{-\alpha|x|}$ for which $\mu(x) = (1-e^{-\alpha|x|})\,\mathrm{sgn}\,x$.

We finally remark that the more general case where the nonlinearity sgn x in Figure 1 is replaced by a, possibly nonmonotonic, general nonlinearity is discussed in [5].

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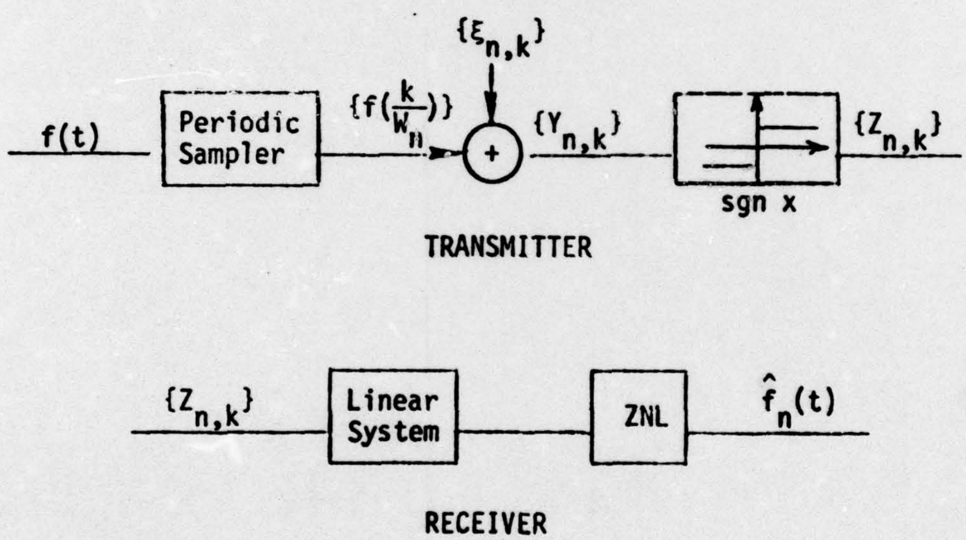

Figure 1. Transmitter/Receiver Structure

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It is well known that a continuous-time signal $f(t)$, $-\infty<t<\infty$, cannot be reconstructed from its 2-level quantized version $\mathrm{sgn}[f(t)]$. It is shown that by deliberately corrupting equally-spaced samples $\{f(k/W)\}$ of f by additive Gaussian noise $\{\xi_k\}$ before hardlimiting, the signal f can be estimated consistently from the binary sequence $\{\mathrm{sgn}[f(k/W) + \xi_k]\}$ as the sampling rate $W\to\infty$. A class of nonlinear estimates is introduced and bounds on the mean-square error are obtained. The signal f need not be bandlimited. 405 248

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